STUDY GUIDE: REAL ANALYSIS

PHIL MAYER

1. Getting Started

Definition 1.1. The **Archimedian Property** of the real numbers says $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x < n$.

Definition 1.2. A set S is **bounded** if and only if $\exists M \in \mathbb{R}$ such that $\forall s \in S$, $|s| \leq M$. We say that a set is **bounded above** if $\exists M \in \mathbb{R}$ such that $s \leq M$, or **bounded below** if $\exists m \in \mathbb{R}$ such that $m \leq s$.

Definition 1.3. A set is **unbounded** if and only if $\forall M \in \mathbb{R}, \exists s \in S$ such that $M < |s|$.

Definition 1.4. A set $S \subset \mathbb{R}$ has **supremum** $\sup(S) = \sup(a, b) = b$ if and only if b is an upper bound for S and b is the *least* of all other upper bounds. So $\forall x \in S, x \leq b$ and $\forall \beta < b\ \exists s \in S$ such that $\beta < s$.

Definition 1.5. We define the **infimum** of a set S similarly: $\forall x \in S, a \leq x$ and $\forall \alpha > a\ \exists s \in S$ such that $a < \alpha$.

Definition 1.6. For some $\epsilon > 0$, define the **ball of radius ϵ around** $x \in \mathbb{R}$ as $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$.

2. Sequences

Definition 2.1. We say $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a **sequence** in the reals. We denote it $\{x_n\}$.

Definition 2.2. $\{x_n\}$ is **monotone increasing** if and only if $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$. Similarly, $\{x_n\}$ is **monotone decreasing** if and only if $\forall n \in \mathbb{N}$, $x_n \geq x_{n+1}$.

Definition 2.3. A sequence $\{x_n\}$ is **bounded** if and only if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |x_n| \leq M$. Bounded above and below can be defined similarly, based on the definition of boundedness of sets.

Definition 2.4. A sequence $\{x_n\}$ **converges** to some limit L if and only if $\forall \epsilon > 0, \exists K \in \mathbb{R}$ such that $\forall n > K, n \in \mathbb{N}, |x_n - L| < \epsilon$.

Definition 2.5. A sequence $\{x_n\}$ does not converge to some limit L if and only if $\exists \epsilon > 0$ such that $\forall K \in \mathbb{R}, \exists n > K$ such that $|x_n - L| \geq \epsilon$.

Definition 2.6. A sequence $\{x_n\}$ is **Cauchy** if and only if $\forall \epsilon > 0, \exists K \in \mathbb{R}$ such that $\forall m, n \in \mathbb{N}$, if $m, n > K$ then $|x_n - x_m| < \epsilon$.

Theorem 2.7 (Squeeze Theorem)**.** *Suppose* $\{a_n\}, \{b_n\}$, *and* $\{c_n\}$ *are sequences where* $\{a_n\}, \{b_n\} \to L$ *and* $\forall n \in \mathbb{N}, a_n \leq b_n \leq c_n$. *Then* $\{b_n\} \to L$.

To prove the Squeeze Theorem, use the convergence of $\{a_n\}$ and $\{c_n\}$ to choose $\epsilon_1 = \epsilon_2 = \epsilon$, then get K_1 so that $\forall n > K_1, |a_n - L| < \epsilon$. This gives us a lower bound on a_n: $L - \epsilon < a_n$. We get K_2 similarly so that $c_n < L + \epsilon$. The $\epsilon/2$ trick may also help here.

Theorem 2.8. *Let* $\{a_n\} \to a$ *and* $\{b_n\} \to b$. *Then* $\{a_n \pm b_n\} \to a \pm b$ *and* $\{a_n b_n\} \to ab$. *If* $\lambda \in \mathbb{R}$, *then* $\{\lambda a_n\} \to \lambda a$. *If* $b_n \neq 0 \, \forall n \in \mathbb{N}$ *and* $b \neq 0$, *then* $\{\frac{a_n}{b_n}\} \to \frac{a}{b}$.

Theorem 2.9. $\{x_n\}$ *converges* $\iff$ $\{x_n\}$ *Cauchy* $\iff$ $\{x_n\}$ *is bounded and monotone.*

Theorem 2.10. *Let* S *be a non-empty set of real numbers which is bounded above. Then* S *has a unique supremum.*

Theorem 2.11 (Bolzano-Weierstrass)**.** *Every bounded sequence* $\{x_n\} \subset \mathbb{R}$ *has a convergent subsequence.*

3. Continuity

Definition 3.1. A function f is **continuous** at $c \in \text{Dom}(f)$ if and only if $\forall \{x_n\} \subset \text{Dom}(f)$ where $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = f(c)$.

Theorem 3.2. *For all powers* $p \in \mathbb{Z}, f(x) = x^p$ *is continuous over its domain.*

Theorem 3.3. *If* f *and* g *are continuous functions, then* $f \pm g, fg$, *and* $f \circ g$ *are continuous on* $\text{Dom}(f) \cap \text{Dom}(g)$. *If* $g(x) \neq 0 \; \forall x \in \text{Dom}(g)$, *then* $\frac{f}{g}$ *is continuous. So any polynomial function is continuous.*

Definition 3.4. f is **discontinuous** at $c \in \text{Dom}(f)$ if and only if $\exists \{x_n\} \subset \text{Dom}(f)$ such that $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Definition 3.5. f is **bounded** if and only if $\exists M \in \mathbb{R}$ such that $\forall x \in \text{Dom}(f), |f(x)| \leq M$.

Definition 3.6. f has limit L at $x = a$, or $\lim_{x\to a} f(x) = L$, if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in \text{Dom}(f)$, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Definition 3.7 ($\epsilon - \delta$)**.** f is **continuous** at $c \in \text{Dom}(f)$ if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in \text{Dom}(f)$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Definition 3.8. f is **uniformly continuous** if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, y \in \text{Dom}(f)$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem 3.9. *If* f *is continuous on a closed, finite interval then* f *is bounded.*

Theorem 3.10 (Extreme Value Theorem)**.** *If* f *is continuous on a closed interval* $[a, b]$, *then* $\exists c \in [a, b]$ *such that* $\forall x \in [a, b], f(x) \leq f(c)$ *and* $\exists d \in [a, b]$ *such that* $\forall x \in [a, b], f(d) \leq f(x)$.

Theorem 3.11 (Intermediate Value Theorem)**.** *Suppose f is continuous on a closed interval $[a,b]$, $f(a) \neq f(b)$, and y is between $f(a)$ and $f(b)$. Then $\exists c \in [a,b]$ such that $f(c) = y$.*

Definition 3.12. If $a \in \mathrm{Dom}(f)$, define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to be the **derivative of** f at $x = a$. If this quantity is defined, f is said to be **differentiable** at $x = a$.

Theorem 3.13 (Rolle's Theorem)**.** *If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b) = 0$, then $\exists c \in (a,b)$ such that $f'(c) = 0$.*

Theorem 3.14 (Mean Value Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 3.15. *If f is continuous on $[a,b]$, $f(c)$ is a maximum where $a < c < b$, and f is differentiable at c, then $f'(c) = 0$.*

Theorem 3.16. *If f is differentiable at $x = a$ then f is continuous at a.*

Theorem 3.17. *If $f(x) = g(x)$ except at $x = a$, then $lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.*

Theorem 3.18 (Product Rule)**.** *If f, g are differentiable at $x = a$ then $f(x)g(x)$ is differentiable at $x = a$ as well. The derivative of the product will be $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

Theorem 3.19 (Power Rule)**.** *If $f(x) = cx^p$ then $f'(x) = (cp)x^{p-1}$.*

4. Introduction to Topology

Definition 4.1. A set $S \subset \mathbb{R}$ is **open** if and only if $\forall x \in S, \exists \epsilon > 0$ such that $B_\epsilon(x) \subset S$.

Definition 4.2. A set $S \subset \mathbb{R}$ is **closed** if and only if $\mathbb{R} \setminus S$ is open.